



**GCE A LEVEL**

1300U30-1A



S24-1300U30-1A

**TUESDAY, 4 JUNE 2024 – MORNING**

**MATHEMATICS – A2 unit 3  
PURE MATHEMATICS B**

**Formula Booklet**

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**GCE AS/A Level MATHEMATICS**

**GCE AS/A Level FURTHER  
MATHEMATICS**

**FORMULA BOOKLET**

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## Pure Mathematics

### Mensuration

$$\text{Surface area of sphere} = 4\pi r^2$$

$$\text{Area of curved surface of cone} = \pi r \times \text{slant height}$$

### Arithmetic Series

$$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$$

### Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \quad \text{for } |r| < 1$$

### Summations

$$\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

### Binomial Series

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n \quad (n \in \mathbb{N})$$

$$\text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r}x^r + \dots \quad (|x| < 1, n \in \mathbb{R})$$

### Logarithms and exponentials

$$e^{x \ln a} = a^x$$

### Complex Numbers

$$\{r(\cos \theta + i \sin \theta)\}^n = r^n (\cos n\theta + i \sin n\theta)$$

The roots of  $z^n = 1$  are given by  $z = e^{\frac{2\pi ki}{n}}$ , for  $k = 0, 1, 2, \dots, n-1$

*Maclaurin's and Taylor's Series*

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^r}{r!} f^{(r)}(0) + \dots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^r}{r!} f^{(r)}(a) + \dots$$

$$f(a+x) = f(a) + xf'(a) + \frac{x^2}{2!} f''(a) + \dots + \frac{x^r}{r!} f^{(r)}(a) + \dots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad (-1 < x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 \leq x \leq 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2r+1}}{(2r+1)!} + \dots \quad \text{for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2r}}{(2r)!} + \dots \quad \text{for all } x$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 < x < 1)$$

*Hyperbolic Functions*

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln \left\{ x + \sqrt{x^2 - 1} \right\} \quad (x \geq 1)$$

$$\sinh^{-1} x = \ln \left\{ x + \sqrt{x^2 + 1} \right\}$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \quad (|x| < 1)$$

### Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \left(A \pm B \neq \left(k + \frac{1}{2}\right)\pi\right)$$

$$\text{For } t = \tan \frac{1}{2}A: \sin A = \frac{2t}{1+t^2}, \quad \cos A = \frac{1-t^2}{1+t^2}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

### Vectors

The resolute of  $\mathbf{a}$  in the direction of  $\mathbf{b}$  is  $\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$ .

The point dividing  $AB$  in the ratio  $\lambda : \mu$  is  $\frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$ .

The equation of a plane in Cartesian form is  $n_1x + n_2y + n_3z = k$ .

The perpendicular distance between two skew lines is  $D = \frac{|(\mathbf{b} - \mathbf{a}) \cdot \mathbf{n}|}{|\mathbf{n}|}$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are position vectors of points on each line and  $\mathbf{n}$  is a mutual perpendicular to both lines.

The perpendicular distance between a point and a line is  $D = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$ , where the coordinates of the point are  $(x_1, y_1)$  and the equation of the line is given by  $ax + by = c$ .

The perpendicular distance between a point and a plane is  $D = \frac{|n_1\alpha + n_2\beta + n_3\gamma - k|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$ , where  $(\alpha, \beta, \gamma)$  are the coordinates of the point and  $n_1x + n_2y + n_3z = k$  is the equation of the plane.

*Matrix transformations in 2-D*

Anticlockwise rotation through  $\theta$  about  $O$ :  $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

Reflection in the line  $y = (\tan \theta)x$ :  $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

*Matrix transformations in 3-D*

Anticlockwise rotation through  $\theta$  about:

<b>x-axis</b>	<b>y-axis</b>	<b>z-axis</b>
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$	$\begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$	$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

where

- an anticlockwise (or positive) rotation about  $Ox$  (or  $x$ -axis) is in the sense  $\mathbf{j} \rightarrow \mathbf{k}$ ,
- an anticlockwise (or positive) rotation about  $Oy$  (or  $y$ -axis) is in the sense  $\mathbf{k} \rightarrow \mathbf{i}$ ,
- an anticlockwise (or positive) rotation about  $Oz$  (or  $z$ -axis) is in the sense  $\mathbf{i} \rightarrow \mathbf{j}$ .

## *Differentiation*

<b>Function</b>	<b>Derivative</b>
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Integration (+ constant;  $a > 0$  where relevant)

Function	Integral
$\tan x$	$\ln \sec x $
$\cot x$	$\ln \sin x $
$\operatorname{cosec} x$	$-\ln \operatorname{cosec} x + \cot x  = \ln\left \tan\left(\frac{1}{2}x\right)\right $
$\sec x$	$\ln \sec x + \tan x  = \ln\left \tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right $
$\sec^2 x$	$\tan x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) \quad ( x  < a)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left\{x + \sqrt{x^2 - a^2}\right\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right) = \ln\left\{x + \sqrt{x^2 + a^2}\right\}$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln\left \frac{a+x}{a-x}\right  = \frac{1}{a} \tanh^{-1}\left(\frac{x}{a}\right) \quad ( x  < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \ln\left \frac{x-a}{x+a}\right $

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Area of a sector

$$A = \frac{1}{2} \int r^2 d\theta \quad (\text{polar coordinates})$$

## Numerical Mathematics

### Numerical integration

The trapezium rule:  $\int_a^b y \, dx \approx \frac{1}{2}h \{(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\}$ , where  $h = \frac{b-a}{n}$

### Numerical Solution of Equations

The Newton-Raphson iteration for solving  $f(x) = 0$ :  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

## Mechanics

### Motion in a circle

Transverse velocity:  $v = r\dot{\theta} = \omega r$

Radial acceleration:  $-r\dot{\theta}^2 = -\frac{v^2}{r} = -\omega^2 r$

### Centres of Mass of Uniform Bodies

Triangular lamina:  $\frac{2}{3}$  along median from vertex

Semi circle:  $\frac{4r}{3\pi}$  from straight edge along axis of symmetry

Quarter circle:  $\bar{x} = \frac{4r}{3\pi}$   $\bar{y} = \frac{4r}{3\pi}$  from vertex

## Probability & Statistics

### Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A)P(B | A)$$

$$P(A | B) = \frac{P(A)P(B | A)}{P(A)P(B | A) + P(A')P(B | A')}$$

$$\text{Bayes' Theorem: } P(A_j | B) = \frac{P(A_j)P(B | A_j)}{\sum P(A_i)P(B | A_i)}$$

### Discrete distributions

For a discrete random variable  $X$  taking values  $x_i$  with probabilities  $p_i$

$$\text{Expectation (mean): } E(X) = \mu = \sum x_i p_i$$

$$\text{Variance: } \text{Var}(X) = \sigma^2 = \sum (x_i - \mu)^2 p_i = \sum x_i^2 p_i - \mu^2$$

$$\text{For a function } g(X): E(g(X)) = \sum g(x_i) p_i$$

Standard discrete distributions:

Distribution of $X$	$P(X = x)$	Mean	Variance
Binomial $B(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	$np$	$np(1-p)$
Poisson $Po(\lambda)$	$e^{-\lambda} \frac{\lambda^x}{x!}$	$\lambda$	$\lambda$

### Continuous distributions

For a continuous random variable  $X$  having probability density function  $f$

$$\text{Expectation (mean): } E(X) = \mu = \int x f(x) dx$$

$$\text{Variance: } \text{Var}(X) = \sigma^2 = \int (x - \mu)^2 f(x) dx = \int x^2 f(x) dx - \mu^2$$

$$\text{For a function } g(X): E(g(X)) = \int g(x) f(x) dx$$

$$\text{Cumulative distribution function: } F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$$

Standard continuous distributions:

Distribution of $X$	P.D.F.	Mean	Variance
Uniform (Rectangular) on $[a, b]$ $U[a, b]$	$\frac{1}{b-a}$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	$\mu$	$\sigma^2$
Exponential $\text{Exp}(\lambda)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

### Expectation algebra

For independent random variables  $X$  and  $Y$

$$E(XY) = E(X)E(Y), \quad \text{Var}(aX \pm bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$$

### Sampling distributions

For a random sample  $X_1, X_2, \dots, X_n$  of  $n$  independent observations from a distribution having mean  $\mu$  and variance  $\sigma^2$

$$\bar{X} \text{ is an unbiased estimator of } \mu, \text{ with } \text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

$$S^2 \text{ is an unbiased estimator of } \sigma^2, \text{ where } S^2 = \frac{\sum (X_i - \bar{X})^2}{n-1}$$

For a random sample of  $n$  observations from  $N(\mu, \sigma^2)$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{(n-1)}$$

If  $X$  is the observed number of successes in  $n$  independent Bernoulli trials in each of which the probability of success is  $p$ , and  $Y = \frac{X}{n}$ , then

$$E(Y) = p \quad \text{and} \quad \text{Var}(Y) = \frac{p(1-p)}{n}$$

For a random sample of  $n_x$  observations from  $N(\mu_x, \sigma_x^2)$  and, independently, a random sample of  $n_y$  observations from  $N(\mu_y, \sigma_y^2)$

$$\frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{\frac{\sigma_x^2}{n_x} + \frac{\sigma_y^2}{n_y}}} \sim N(0,1)$$

## Correlation and Regression

For a sample of  $n$  pairs of observations  $(x_i, y_i)$

$$S_{xx} = \sum (x_i - \bar{x})^2 = \sum x_i^2 - \frac{(\sum x_i)^2}{n}$$

$$S_{yy} = \sum (y_i - \bar{y})^2 = \sum y_i^2 - \frac{(\sum y_i)^2}{n}$$

$$S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \frac{(\sum x_i)(\sum y_i)}{n}$$

A measure of linear association between two variables  $X$  and  $Y$  is given by the Pearson product - moment correlation coefficient  $r$ .

For the sample  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , it is given by  $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$ .

Given data, the parameters  $\alpha$  and  $\beta$  of the linear regression model may be estimated using the principle of least squares.

The least squares estimate  $\hat{\beta}$  of the parameter  $\beta$  is given by  $\hat{\beta} = \frac{S_{xy}}{S_{xx}}$ .

The least squares estimate  $\hat{\alpha}$  of the parameter  $\alpha$  is given by  $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$ .

The least squares regression line is given by  $y = \hat{\alpha} + \hat{\beta}x$ .

Spearman's rank correlation coefficient is given by  $r_s = 1 - \frac{6\sum d_i^2}{n(n^2 - 1)}$ .

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